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EVALUATION OF THE PARAMETRIC INSTABILITY OF AN AXIALLY TRANSLATING MEDIA USING A VARIATIONAL PRINCIPLE

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1. INTRODUCTION

The dynamic instability problem of translating media covering band-saws, paper webs, transmission belts, oil pipelines and exhaust mufflers has been of academic and engineering interest. The major concern is the determination of boundaries separating stability and instability. When the translating velocity or tension is constant, the dynamic behavior of such translating media is governed by a partial differential equation with constant coefficients. However, the coefficients of the partial differential equation will become time-dependent periodic when the translating velocity or tension P has a time-dependent periodic component superposed on a constant, $P = P_0 + \alpha P_1(t)$, where $P_1(t)$ is a periodic function of time t with its period T_0, and P_0 and α are constants. For the partial differential equations with time-dependent periodic coefficients, parametric instability has to be considered.

The parametric instability of axially moving media with a periodic tension force has been investigated by some researchers. Wickert and Mote [1] reviewed the dynamic behavior of moving materials with a focus on moving strings and beams. Paidoussis and Li [2] presented a survey on the dynamic instability of pipes conveying fluid. To [3] investigated the acoustic pulsation of mufflers and pipelines with pulsating flow velocity. Mote [4] evaluated the parametric instability boundary of a translating string using numerical methods that involve the replacement of the spatial derivatives with finite differences and integration of the set of Mathieu equations. Wu and Mote [5] also investigated the parametric excitation of an axially moving band under periodic loading. Mockensturm *et al.* [6] addressed the issue of stability and limit cycles of moving strings parametrically excited. Pakdemirli and Ulsoy [7] investigated the stability of an axially accelerating string. Recently, Parker and Lin [8] studied the parametric instability of axially moving media subjected to multi-frequency tension and speed fluctuations.

The available methods for the evaluation of parametric instability boundaries of such problems are the perturbation method (also called the small parameter methods), for example, reference [9]; and the Galerkin method. The perturbation method is comparatively simple in its concept; but it can only determine a portion of stability boundaries. The Galerkin method can reduce the partial differential equation with periodic coefficients into a set of ordinary differential equations with periodic coefficients; then this set of ordinary differential equations with periodic coefficients can be either cast into an eigenvalue problem using Bolotin's method [10] or solved using the perturbation methods. The methods mentioned above sometimes give poor evaluation accuracy. In

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addition, the Galerkin method requires that trial functions be comparison functions, satisfying both essential boundary conditions and natural boundary conditions. Such requirements may bring about difficulties in choosing trial functions if the system's layout involves free ends, in-span supports or connection to other spring–mass systems. In past practice, the eigensolutions of the corresponding stationary system were often used as a set of trial functions of the Galerkin method. But, the eigensolutions of the stationary system may violate the natural boundary conditions of the axially translating system and thus the evaluation accuracy is in doubt. For two-dimensional problems like wide paper webs and wide band saws, it would be very challenging to implement the Galerkin method because of the difficulty in choosing proper admissible functions.

In this paper, a variational method is used for the formulation of the dynamic instability assessment of axially translating media; and the parametric instability problem is reduced into a stationary value problem in the form of a classical Rayleigh quotient. As a result, the trial functions do not have to satisfy the natural boundary conditions. Moreover, the approximation error of the functional, resulting from series truncation, is of the second order if the error of the trial functions is of the first order.

In the last part of the paper, the example of an axially translating string is presented to demonstrate the variational principle proposed in this paper. The effect of gyroscopic force on the stability region boundary is also examined using this variational principle with the stability region boundary obtained by Mote [4] as a cross-check on the predictions by the present method. The aim of this study is the establishment of a basis for the numerical solution of such problems using Rayleigh–Ritz methods, and in particular, finite element methods.

2. THE VARIATIONAL FORMULATION FOR AN AXIALLY TRANSLATING STRING AND BEAM

The linerized equation of transverse motion for the axially moving string with a periodic tension component is (see reference [4])

$$\rho \frac{\partial^2 w}{\partial t^2} + 2V\rho \frac{\partial^2 w}{\partial x \partial t} + \rho V^2 \frac{\partial^2 w}{\partial x^2} - (P_0 + \alpha P_1(t)) \frac{\partial^2 w}{\partial x^2} = 0 \tag{1}$$

with its essential boundary conditions given by

$$w(0,t) = w(l,t) = 0. \tag{2}$$

where V is the axial transport velocity; $w(x,t)$is the transverse displacement; t is the time; x is the length co-ordinate; P_0 is the mean value of the tension in one period; $\alpha P_1(t)$ is the periodic time-varying component of the tension with T_0 as its period; and α is the scalar variable.

The parametric instability problem can be restated in this way: to find the value of the scalar variable α so that there exists a non-trivial periodic solution $w(x,t)$ $((w(x,t) = w(x,t+T))$ that satisfies equation (1) and the essential boundary condition (2). Note that the period T of $w(x,t)$ can be integer multiples of the period T_0 of $P_1(t)$. For the principal parametric instability, T is twice as large as T_0, i.e., $T = 2T_0$. It can be seen from this statement that the dynamic instability problem is a typical eigenvalue problem. Therefore, we attempt to reformulate this parametric instability problem using a concept similar to that of the classical Rayleigh quotient. To do so, one has to find the associated functional, the quotient, governing the transverse motion of the axially translating string. Following the treatment of the dynamic instability of a column by Hu [11, 12], such functional can be established as

$$\alpha = \frac{\Pi_1}{\Pi_2}, \tag{3}$$

where

$$\Pi_1 = \int_0^T \int_0^l \left[(\rho V^2 - P_0) \left(\frac{\partial w}{\partial x} \right)^2 + \rho \left(\frac{\partial w}{\partial t} \right)^2 + 2\rho V \frac{\partial w \partial w}{\partial x\, \partial t} \right] \mathrm{d}x\, \mathrm{d}t \tag{3a}$$

and

$$\Pi_2 = \int_0^T \int_0^l P(t) \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}x \mathrm{d}t. \tag{3b}$$

Equations (3, 3a, 3b) indicate that α is a functional of $w(x,t)$. In can be proved that α achieves its stationary value when $w(x,t)$ is the true solution. That is to say, the true solution $w(x,t)$, that satisfies equation (1), boundary condition equation (2) and the periodicity condition $(w(x,t) = w(x,t+T))$, will provide a stationary value for the functional, defined by equation (3),

$$\delta\alpha = 0; \tag{4}$$

here $\delta(\cdot)$ denotes the operation of taking $(\cdot)$ first order variation with respect to $w(x,t)$.

Following the above procedure one can establish the corresponding variational principle for an axially translating beam.

The linearized equation of the transverse motion for an axially moving Euler–Bernoulli beam with a periodic tension component is

$$\rho \frac{\partial^2 w}{\partial t^2} + 2\rho V \frac{\partial^2 w}{\partial x \partial t} + \rho V^2 \frac{\partial^2 w}{\partial x^2} - (P_0 + \alpha P(t)) \frac{\partial^2 w}{\partial x^2} + EJ \frac{\partial^4 w}{\partial x^4} = 0 \tag{5}$$

with its boundary conditions given by

$$w = 0 \text{ and } \frac{\partial w}{\partial x} = 0 \text{ at fixed ends,} \tag{5a}$$

$$w = 0, \text{ and } EJ \frac{\partial^2 w}{\partial x^2} = 0 \text{ at simply supported ends,} \tag{5b}$$

where EJ is the flexural rigidity of the beam, and the definitions of the other symbols in equations (5, 5a, 5b) are the same as those of the string mentioned above.

The functional associated with equations (5, 5a, 5b) can be written as

$$\alpha = \frac{\Pi_1 - \int_0^T \int_0^l EJ (\partial^2 w / \partial x^2)^2 \,\mathrm{d}x\, \mathrm{d}t}{\Pi_2}, \tag{6}$$

where Π_1 and Π_2 share the same expressions as in equations (3a, 3b). In can be proven that the stationary value problem of the functional defined in equation (6) is equivalent to the original problem.

3. GENERALIZATION

Examining the terms in equations (3, 3a, 3b), one can see that every term has its own physical meaning. Each term represents the average generalized energy in one period of time T, either average generalized potential energy or average generalized kinetic energy. The entire equation means the conservation of average generalized energy in one period of time T, i.e., the average generalized kinetic energy $(KE_{ce} + KE_w + KE_{co})$ is equal to the average generalized potential energy $(PE_0 + PE_b + \alpha PE_p)$,

$$(KE_{ce} + KE_w + KE_{co}) - (PE_0 + PE_b + \alpha PE_p) = 0, \tag{7}$$

where

$$KE_{ce} = \frac{1}{T}\int_0^T \int_0^l \rho V^2 \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t, \; KE_w = \frac{1}{T}\int_0^T \int_0^l \rho \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t \qquad (7\mathrm{a,b})$$

$$KE_{co} = \frac{1}{T}\int_0^T \int_0^l 2\rho V \frac{\partial w}{\partial x}\frac{\partial w}{\partial t} \mathrm{d}x\,\mathrm{d}t, \; PE_b = \frac{1}{T}\int_0^T \int_0^l EJ \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t, \qquad (7\mathrm{c,d})$$

$$PE_p = \frac{1}{T}\int_0^T \int_0^l P(t) \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t, \; PE_0 = \frac{1}{T}\int_0^T \int_0^l P_0 \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t. \qquad (7\mathrm{e,f})$$

Based on the statement of the physical meanings given in equation (7) one can conceive the functional

$$\alpha = \frac{(KE_{ce} + KE_w + KE_{co}) - (PE_0 + PE_b)}{PE_p}. \tag{8}$$

It is not difficult to verify that the original equation of motion plus the natural boundary conditions can be derived by invoking the stationarity of the functional with respect to $w(x, t)$, i.e., $\delta\alpha = 0$, for any particular system. Let us illustrate this using the following two examples.

3.1. EXAMPLE 1

Consider the same translating beam addressed above except that a spring with stiffness coefficient k is connected to the beam at $x = b$, as shown in Figure 1. In addition to all of the energy terms given in equations (3, 3a, 3b), there is one more potential energy term resulting from the spring deformation, it is

$$PE_k = \frac{1}{T}\int_0^T k w^2(b, t)\,\mathrm{d}t. \tag{9}$$

Thus, the associated functional is

$$\alpha = \frac{(KE_{ce} + KE_w + KE_{co}) - (PE_0 + PE_b + PE_k)}{PE_p}$$
$$= \frac{\int_0^T \int_0^l \left[\rho V^2 \left(\frac{\partial w}{\partial x}\right)^2 + \rho \left(\frac{\partial w}{\partial t}\right)^2 + 2\rho V \frac{\partial w}{\partial x}\frac{\partial w}{\partial t} - P_0 \left(\frac{\partial w}{\partial x}\right)^2 - EJ \left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] \mathrm{d}x\,\mathrm{d}t - \int_0^T k w^2(b, t)\,\mathrm{d}t}{\int_0^T \int_0^l P(t) \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t}. \tag{10}$$

and it can be shown that Euler equations of equation (10) are exactly the same as its differential equation plus the natural boundary conditions.

3.2. EXAMPLE 2

Another example is the same beam as discussed above but the spring is replaced with an in-span support, a simply support, at $x = b$, as shown in Figure 2. Combining the generalized variational principle illustrated in Liu *et al.* [13] and the variational principle

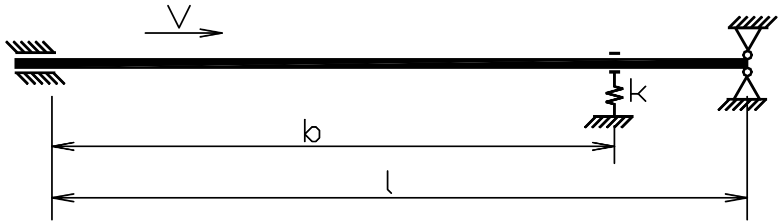

Figure 1. Translating beam under periodic tension with a spring support.

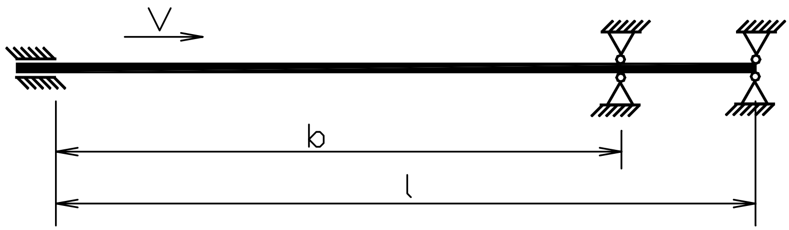

Figure 2. Translating beam under periodic tension with an in-span simple support.

given in equation (7), one can conceive the functional associated with this travelling beam with an in-span support as

$$\alpha = \frac{\int_0^T \int_0^l \left[\rho V^2 \left(\frac{\partial w}{\partial x} \right)^2 + \rho \left(\frac{\partial w}{\partial t} \right)^2 + 2\rho V \frac{\partial w}{\partial x} \frac{\partial w}{\partial t} - P_0 \left(\frac{\partial w}{\partial x} \right)^2 - EJ \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] \mathrm{d}x\, \mathrm{d}t - \int_0^T 2\lambda w(b,t)\, \mathrm{d}t}{\int_0^T \int_0^l P(t) \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}x\, \mathrm{d}t}, \tag{11}$$

where λ is a Lagrange multiplier as an independent argument variable. Invoking the stationarity of this functional α given in equation (11) with respect to both $w(x,t)$and λ, one can derive the equation of motion plus the equation $w(b,t) = 0$ as well as the natural boundary conditions.

Note that the functional given by equation (8) may not be the only functional available to serve such purpose. Other conserved functionals, such as those functionals addressed by Renshaw *et al.* [14], may also be used to develop the variational formulation.

4. THE RAYLEIGH–RITZ METHOD

Using the variational principle established above, one can use either Rayleigh–Ritz methods or finite element methods to transform the variational equation into an algebraic eigenvalue problem. Next, we will illustrate this using the Rayleigh–Ritz method.

In the Rayleigh–Ritz method, $w(x,t)$ is expressed as

$$w(x,t) = \sum_{i=1}^{N} a_i \varphi_i(x,t) = \Phi^{\mathrm{T}} \beta \tag{12}$$

where a_i is the coefficient to be determined and $\varphi_i(x,t)$ is the chosen trial function satisfying all of the essential boundary conditions and the periodicity condition as well. Denote $\Phi^{\mathrm{T}} = [\varphi_1, \varphi_1, \ldots, \varphi_N]$ and $\beta^{\mathrm{T}} = [a_1, a_1, \ldots, a_N]$. It should be noted that the Rayleigh–Ritz method does not require $\varphi_i(x,t)$ to satisfy the natural boundary conditions.

Introduction of equation (12) into the variational expression, for example, equation (6), yields

$$\alpha = \frac{\beta^{\mathrm{T}}(M_{ce} + M_w + M_{co})\beta - \beta^{\mathrm{T}}(K_0 + K_b)]\beta}{\beta^{\mathrm{T}} K_p \beta}, \tag{13}$$

where M_{ce}, M_w, M_{co}, K_o, K_b and K_p are $N \times N$ order matrices,

$$M_{ce} = \int_0^T \int_0^l \rho V^2 \frac{\partial \Phi}{\partial x} \frac{\partial \Phi^{\mathrm{T}}}{\partial x} \mathrm{d}x\,\mathrm{d}t, \quad M_w = \int_0^T \int_0^l \frac{\partial \Phi}{\partial t} \frac{\partial \Phi^{\mathrm{T}}}{\partial t} \mathrm{d}x\,\mathrm{d}t, \tag{13a, b}$$

$$M_{co} = \int_0^T \int_0^l 2\rho V \frac{\partial \Phi}{\partial t} \frac{\partial \Phi^{\mathrm{T}}}{\partial x} \mathrm{d}x\,\mathrm{d}t, \quad K_0 = \int_0^T \int_0^l P_0 \frac{\partial \Phi}{\partial x} \frac{\partial \Phi^{\mathrm{T}}}{\partial x} \mathrm{d}x\,\mathrm{d}t, \tag{13c, d}$$

$$K_b = \int_0^T \int_0^l EJ \frac{\partial \Phi}{\partial x} \frac{\partial \Phi^{\mathrm{T}}}{\partial x} \mathrm{d}x\,\mathrm{d}t, \quad K_p = \int_0^T \int_0^l P(t) \frac{\partial \Phi}{\partial x} \frac{\partial \Phi^{\mathrm{T}}}{\partial x} \mathrm{d}x\,\mathrm{d}t. \tag{13e, f}$$

Taking the first order variation of α with respect to β and making the first order variation vanish, $\delta\alpha = 0$, one obtains the algebraic eigenvalue problem

$$[(M_{ce} + M_w + M_{co}) - (K_0 + K_b)]\beta = \alpha K_p \beta. \tag{14}$$

Now, we have transformed the parametric instability problem into a standard algebraic eigenvalue problem, which can be easily solved using conventional methods.

It should be noted that the symmetry of the matrices M_{ce}, M_w, K_0, K_b and K_p are ensured except M_{co}, which reflects the effect of the gyroscopic force.

5. AN ILLUSTRATING EXAMPLE

For the purpose of illustrating the variational principle proposed in this paper, let us evaluate the parametric instability boundary of the translating string with a tension variation superposed on a mean tension. The parametric instability boundary of such a moving string was evaluated by Mote [4], where the parametric instability boundary was obtained using the numerical methods involving the numerical solution of a set of coupled Mathieu equations obtained by replacing the spatial derivatives with equivalent difference expression. The comparison between the instability boundary obtained by the present method and the one from Mote [4] will be made as a cross-check. In addition, we will use the present method to examine the effect of the gyroscopic forces on the instability boundary.

The non-dimensional linearized equation of transverse motion of the axially translating string is (see reference [4])

$$v_{\tau\tau} + 2cv_{\xi\tau} + (c-1)v_{\xi\xi} - \alpha(\cos 2\pi\theta\tau)v_{\xi\xi} = 0 \tag{15}$$

with its essential boundary conditions given by

$$v(0,\tau) = v(1,\tau) = 0, \tag{16}$$

where v is the non-dimensional transverse displacement; c is the non-dimensional axial transport velocity; α is the non-dimensional tension variation; ξ is the non-dimensional length; τ is the non-dimensional time; and θ is the non-dimensional frequency of the tension variation.

Based on the functional given by equations (3, 3a, 3b), one can obtain the functional associated with this non-dimensional string as

$$\alpha = \frac{\int_0^T \int_0^1 \{(c^2 - 1)(\partial v/\partial \xi)^2 + 2c(\partial v/\partial \xi)(\partial v/\partial \tau) + (\partial v/\partial \tau)^2\}\, \mathrm{d}\xi\, \mathrm{d}\tau}{\int_0^T \int_0^1 \cos 2\pi\theta t (\partial v/\partial \xi)^2\, \mathrm{d}\xi\, \mathrm{d}\tau}. \tag{17}$$

Now the instability boundary problem is transformed into the stationary value problem of the functional α defined by equation (17). Here, we only address the application of the classical Rayleigh–Ritz method to the solution of this stationary value problem. The key to the success of the classical Rayleigh–Ritz method lies in the proper choice of trial functions. The requirements of the trial functions are that (1) they satisfy the essential boundary condition $v(0, \tau) = v(1, \tau) = 0$ and (2) they satisfy the periodicity condition $v(\xi, \tau) = v(\xi, \tau + 2/\theta)$. As can be seen in equation (15) that the period of the tension variation is $1/\theta$; and thus the trial functions are required to have the period $2/\theta$ for the principle parametric instability, i.e., $T = 2/\theta$.

Regarding how to choose the trial functions, much attention has to be given to the gyroscopic force term $\int_0^T \int_0^1 2c(\partial v/\partial \xi)(\partial v/\partial \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau$ in equation (17). When the translating velocity c is quite small, for example, $c \leqslant 0{\cdot}2$, this gyroscopic force term can be ignored and thus the classical normal mode shapes exist; accordingly, it is reasonable to choose the classical normal mode shapes as the trial functions. However, this gyroscopic force term cannot be ignored when the translating speed c is not small, for example, $c \geqslant 0{\cdot}3$. In this case, no classical normal mode shapes exist (see reference [15]). Therefore, it will not lead to an accurate instability boundary if all of the trial functions are in the classical mode shapes.

5.1. CASE 1

First, let us assume that the gyroscopic force term in equation (17) is small enough to be ignored. In this case, the solution $v(\xi, \tau)$ can be approximated using the linear combination of classical mode shapes,

$$v(\xi, \tau) = \sin \pi\xi (a_1 \cos \pi\theta\tau + a_2 \cos 3\pi\theta\tau + a_3 \cos 5\pi\theta\tau), \tag{18}$$

where a_1, a_2 and a_3 are the coefficients to be determined. It is clear that $v(\xi, \tau)$ satisfies the essential boundary condition and the periodicity condition. Introduction of equation (18) into the variational expression (17) with ignoring the gyroscopic force term, $\int_0^T \int_0^1 2c(\partial v/\partial \xi)(\partial v/\partial \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau$, yields,

$$\alpha = \frac{(c^2 - 1)(a_1^2 + a_2^2 + a_3^2) + (a_1^2 + 9a_2^2 + 25a_3^2)\theta^2}{\frac{1}{2}a_1^2 + a_1 a_2 + a_2 a_3}. \tag{19}$$

Invoking the stationarity of α with respect to a_i $(i = 1, 2, 3)$, i.e.,

$$\frac{\partial \alpha}{\partial a_i} = 0, \quad (i = 1, 2, 3).$$

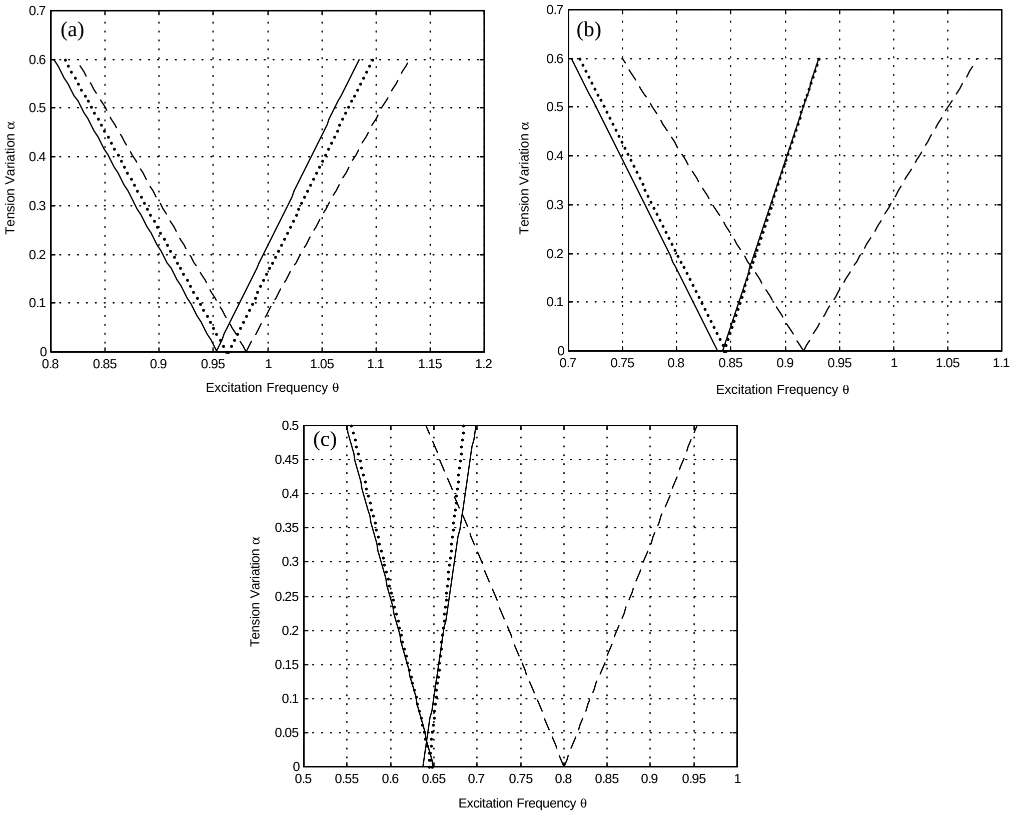

Figure 3. Principal region of dynamic instability for transport velocities (a) $c = 0{\cdot}2$, (b) $c = 0{\cdot}4$, (c) $c = 0{\cdot}6$. Solid lines denote the results presented by Mote [4]; dash lines denote results when the effect of gyroscopic force term is completely ignored and thus the trial functions given by equation (18); dot lines denote the results with the effect of gyroscopic force term considered and the trial functions given by equation (21).

one gets the eigenvalue problem

$$\begin{bmatrix} c^2-1+\theta^2 & 0 & 0 \\ 0 & c^2-1+9\theta^2 & 0 \\ 0 & 0 & c^2-1+25\theta^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \alpha \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{20}$$

For the given set of c and θ values, the eigenvalues α and the associated eigenvectors $[a_1, a_2, a_2]$ can be evaluated by solving this eigenvalue problem; and the instability boundary is plotted in Figures 3(a–c) using dash and lines. For the purpose of comparison, the instability boundary evaluated by Mote [4] is also plotted using solid lines in Figures 3(a–c).

It can be seen from Figures 3(a–c) that the instability boundary given by the present method is quite close to the one given by Mote [4] when the translating velocity c is smaller than $0{\cdot}2$. The larger the translating velocity c, the more significant the difference between the instability boundary given by the present method and the one in reference [4]. This difference is due to the fact that the gyroscopic force term, $\int_0^T \int_0^1 2c(\partial v/\partial \xi)(\partial v/\partial \tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$, has been taken out in the functional defined in equation (17). In the next part, we are going to take account of this gyroscopic term and its effect on the evaluation accuracy of the instability boundary will be shown. Regarding the eigenvectors, i.e., instability mode shape, it can be observed that a_3 is much smaller than a_1 and a_2. This means that the first two terms of trial functions alone can give satisfactory accuracy.

Note that the gyroscopic force term is equal to zero, i.e., $\int_0^T \int_0^1 2c(\partial v/\partial \xi)(\partial v/\partial \tau)\,\mathrm{d}\xi\,\mathrm{d}\tau = 0$, if the trial functions have the classical mode shapes like those given in equation (18). In other words, the gyroscopic force term will not bring any effect to the eigenvalues and the eigenvectors if the classical mode shapes are the exclusive kind of trial functions.

5.2. CASE 2

As pointed in the book by Chen [15], no classical normal mode shape exists because of the gyroscopic force term. Chen [15] argued that the motions associated with the symmetric mode shapes will produce a gyroscopic force, which, in turn, will induce motion in the anti-symmetric mode shapes. This means that trial functions must include, besides the classical mode shapes, those trial functions which can take the effect of the gyroscopic force into account. To handle this, Wickert and Mote [16] tried complex mode shapes. With this point in mind, we choose the trial functions in the following form:

$$\begin{aligned} v(\xi,\tau) = {} & \sin(\pi\xi)(a_1\cos\pi\theta\tau + a_2\cos 3\pi\theta\tau) \\ & + a_3 x(x-1)\sin(\pi\xi + \pi\theta\tau) + a_4 x(x-1)\sin(\pi\xi - \pi\theta\tau) \\ & + a_5 x(x-1)\sin(\pi\xi + 3\pi\theta\tau) + a_6 x(x-1)\sin(\pi\xi - 3\pi\theta\tau), \end{aligned} \tag{21}$$

where $\sin(\pi\xi)(a_1\cos\pi\theta\tau + a_2\cos 3\pi\theta\tau)$ has the form of classical normal mode shapes; $a_3x(x-1)\sin(\pi\xi+\pi\theta\tau) + a_4x(x-1)\sin(\pi\xi-\pi\theta\tau)$ is a couple of waves travelling in opposite directions; $a_5x(x-1)\sin(\pi\xi+3\pi\theta\tau) + a_6x(x-1)\sin(\pi\xi-3\pi\theta\tau)$ also represents a couple of waves travelling in opposite directions but with different wave velocity; a_i $(i = 1, 2, 3, \ldots, 6)$ is the coefficient to be determined through invoking the stationarity of the functional with respect to a_i. The reason for choosing these two sets of waves as the components of the trial functions is to try to reflect the effect of the gyroscopic force term because the gyroscopic force has the tendency to move the mode shapes.

Through the introduction of equation (21) into equation (17), and by invoking the stationarity of α with respect to a_i $(i = 1, 2, 3, \ldots, 6)$, one can get the eigenvalue problem with α as the eigenvalue and $[a_1, a_2, a_3, a_4, a_5, a_6]$ as the eigenvector.

For the given set of c and θ values, the eigenvalues α and the associated eigenvectors $[a_1, a_2, a_3, a_4, a_5, a_6]$ are evaluated by solving this eigenvalue problem; and the instability boundary is plotted in Figures 3(a–c) by dash-dot lines.

It can be seen from Figures 3(a–c) that the instability boundary, associated with this set of trial functions, is very close to the one given by Mote [4] even when the translating velocity c reaches 0·6. Note that the gyroscopic force term is not zero anymore, $\int_0^T \int_0^1 2c(\partial v/\partial \xi)(\partial v/\partial \tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \neq 0$, for the trial functions given by equation (21). In other words, the trial functions given by equation (21) indeed reflect the effect of the gyroscopic force term. That is why the instability boundary evaluated by the trial functions defined by equation (21) retains its better accuracy even when the translating velocity c is as large as 0·6. In contrast, the instability boundary obtained by the classical mode shapes given in equation (18) is not accurate at all for the translating velocity c greater than 0·3. The reason for this is that the classical mode shapes do not cover the effect of the gyroscopic force term even though the gyroscopic force term is significant when the translating velocity c becomes greater than 0·3.

It is also quite interesting to see the trend of the eigenvectors with respect to the translating velocity, c. In can be seen that a_1 and a_2 decrease but a_3, a_4, a_5, and a_6 increase when the translating velocity, c, increases. Note that a_1, and a_2 are the coefficients of the classical mode shapes, $\sin(\pi\xi)(a_1 \cos \pi\theta\tau + a_2 \cos 3\pi\theta\tau)$; while a_3, a_4, a_5, and a_6 are the coefficients of the waves, $x(x-1)[a_3 \sin \pi(\xi + \theta\tau) + a_4 \sin \pi(\xi - \theta\tau) + a_5 \sin \pi(\xi + 3\theta\tau) a_6 \sin \pi(\xi - 3\theta\tau)]$. This trend demonstrates that the effect of the gyroscopic force term increases with the increase of translating velocity, c.

6. CONCLUDING REMARKS

This paper proposed the variational principle for parametric instability analysis of axially translating media. The functional associated with the variational principle is similar to the classical Rayleigh quotient in that it represents the conservation of the average generalized kinetic energy and the average generalized potential energy. After realizing the fact that the average generalized energy is conservative, it would not be difficult to derive the variational principle given in this paper via application of Hamilton's Principle. Since it is formulated as a stationary value problem, the error of eigenvalues is in the second order when the error of the Rayleigh–Ritz trial functions is in the first order. In addition, the current method does not require the trial functions to satisfy the natural boundary conditions and as a result, it brings about simplicity in choosing trial functions compared with the Galerkin methods. Moreover, the method proposed here can generate the eigenfunctions associated with the eigenvalues. These eigenfunctions provide very useful information about the spatial–temporal shape of the translating media when it is in the critical state. Finally, the variational principle proposed here offers the foundation for approximate solution of such problems using finite element methods. For two-dimensional structures like wide paper webs and wide saw blades, the finite element methods would be promising with the variational principle proposed in the paper as its theoretical basis. The numerical example shows that two or three terms of trial functions will give a very accurate instability boundary using the classical mode shapes as the trial functions when the translating velocity $c \leqslant 0{\cdot}2$. When $c \geqslant 0{\cdot}2$, the effect of the gyroscopic force term is significant and thus the set of trial functions must include non-classical mode shapes. The combination of the classical mode shapes and the two sets of waves do lead to a very accurate instability boundary even when $c = 0{\cdot}6$.

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